Acyclic Orientations and Chromatic Polynomials

Adam Deng

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Abstract

This paper revolves around insights on graph colorings and acyclic orientations as introduced by Richard P. Stanley in his 1973 paper. We motivate the subject with a famous problem, then introduce preliminaries. The main content is two theorems, which will be fully proved in this paper. The first relates acyclic orientations on a graph to the chromatic polynomial evaluation at -1. The second is an equality on the number of labeled acyclic digraphs on n-vertex graphs. Finally, further work will be displayed. The paper is intended for undergraduates and will assume some knowledge of graph theory.

1 Background and Motivation

We start with a famous problem.

Four Color Question. We are given a 2-D space divided into shapes. What is the minimum number of colors necessary to paint all the shapes such that no two bordering shapes (diagonals-only do not count as adjacent) have the same color?

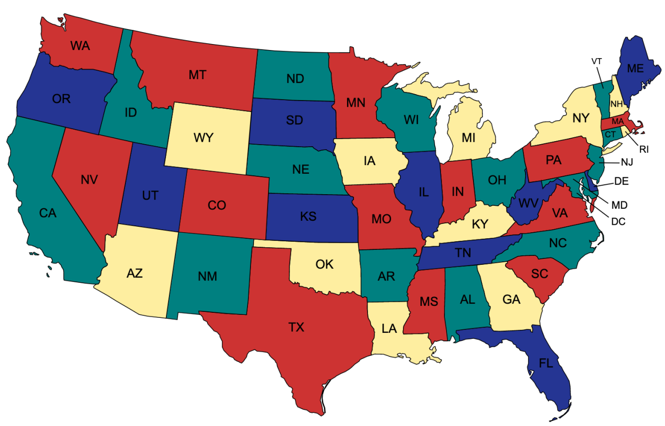

Figure 1: A map of the contiguous United States, created using [Map23], which paints each of the 48 states with one of 4 colors, such that no bordering states share the same color.

In Fig. 1 we see a contiguous US map with four colors. In fact, we can prove that, for any 2-D surface partitioned into shapes, we need at most four colors. This theorem was famously demonstrated by Kenneth Appel and Wolfgang Haken in 1976 in [AH78] through algorithmic methods.

A helpful breakthrough in proving conjectures on minimum number of colors required is to swap the map for an equivalent graph. Specifically, each shape is a vertex, and an edge is connected between two vertices if and only if the two shapes share a border. In this way, whether two shapes can be colored the same becomes a question of whether two vertices are adjacent (i.e. have an edge between them), which is far easier to manage.

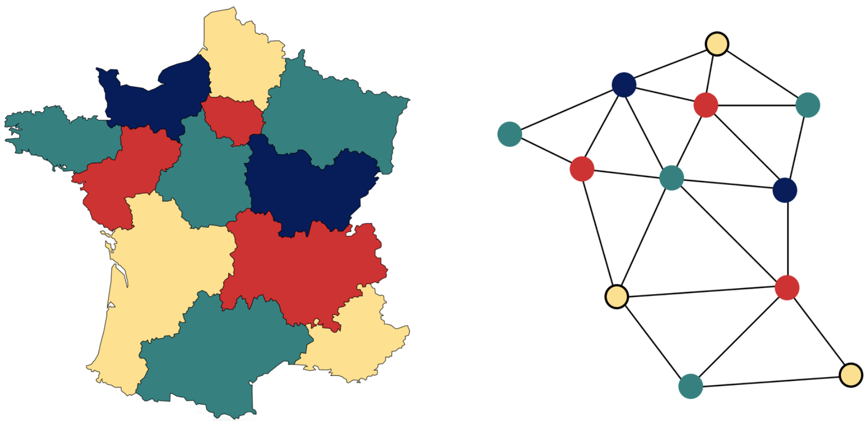

Figure 2: Left: A map of the regions of mainland France, using [Map23]. Four colors are necessary and sufficient to color each region so that no bordering regions share the same color. Right: The same map, using a graph instead. Vertices are colored according to their coloring on the France map.

2 Preliminaries and Context

Now that we can represent any map as a graph G, we formalize the notation with the following concept:

The **chromatic number** of a graph G is defined as the minimum number of colors λ^* necessary such that each vertex of G can be colored with one of the colors $\{1, 2, \ldots, \lambda^*\}$ with no two adjacent vertices the same color.

Example We can see that the chromatic number of the map of the contiguous United States (or, more accurately, its equivalent graph) is 4. To see that it is no lower, consider Nevada. It has an odd number of neighbors. Suppose for contradiction that the US map could be colored with 3 colors (red, blue, yellow) instead. Without loss of generality, let Nevada be yellow and California be blue. Then, Oregon must be red, Idaho blue, Utah red, and Arizona blue to avoid bordering states' having the same colors. But then California and Arizona, which border, have the same color, contradiction. Thus, the US map requires 4 colors, and as we saw in Fig. 1, a valid construction was made.

Observation More generally, if in graph G there exists a vertex v connected to each of vertices $v_1, v_2, \ldots, v_n$ with n odd and $v_1, v_2, \ldots, v_n$ forming an odd-length cycle, at least four colors are necessary to color G.

Afterwards, we may consider, given a certain number of colors λ to color the vertices of G with, how many *ways* there are to do so.

Define the **chromatic polynomial** of a graph G, χ_G, as the unique polynomial so that $\chi_G(\lambda)$ (the subscript is sometimes omitted for convenience) equals the number of ways G's vertices can be colored with one of the λ colors such that no two adjacent vertices have the same color.

Example For the clique on 4 vertices K_4, $\chi(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)$. Label them A, B, C, D, in clockwise order. Select any vertex, WLOG B. There are λ ways to color it, as no restrictions are applied. Then consider D; it can be one of $\lambda - 1$ colors: any except the color of B. Apply this logic to C and A, which have $\lambda - 2$ and $\lambda - 3$ options, respectively. We could also have determined the number of remaining feasible colors in a different order, say, D, A, B, C. The answer is the same. See Fig. 3 for a depiction of this scenario.

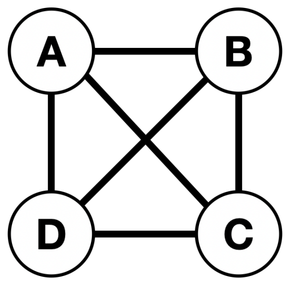

Figure 3: The clique K_4, with labeled vertices.

Observation If the chromatic number of G is λ^*, then for all $0 \leq \lambda < \lambda^*$, $\chi_G(\lambda) = 0$. In other words, there are 0 ways to color G's vertices with fewer colors than necessary.

Example For K_4, whose chromatic number is 4 (because every vertex must have a different color), $\chi_{K_4}(0) = \chi_{K_4}(1) = \chi_{K_4}(2) = \chi_{K_4}(3) = 0$.

Observation $\chi_G(\lambda)$ is not necessarily equal to 0 for negative λ. For K_4, $\chi_{K_4}(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)$ at $\lambda = -1$ actually yields $-1 \cdot (-2) \cdot (-3) \cdot (-4) = 24$. Note that χ is defined for negative λ! This will be important in coming sections.

In the early 1970s, chromatic polynomials, a new venture in themselves, represented one of many directions of study in graph theory. Richard P. Stanley, in [Sta06], studied the relationships between chromatic polynomials and other familiar concepts in graph theory, and derived several surprising results. These unexpected connections allow for easy calculation of certain quantities, which will be discussed in detail in the following sections.

A **directed graph**, or digraph, is a graph G for which every edge in G is assigned a direction; in other words, the edge between v_1 and v_2 either points (with an arrow) from v_1 towards v_2 (denoted $v_1 \to v_2$) or from v_2 towards v_1 (denoted $v_2 \to v_1$). Fig. 4 is a directed graph.

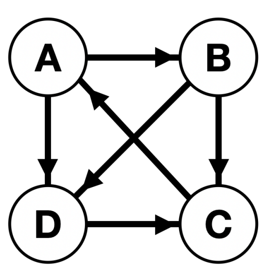

Figure 4: The clique K_4, with labeled vertices and directed edges. This is an orientation, but not an acyclic orientation ($A \to B \to C \to A$ forms a cycle).

An **orientation** of a graph G is an assignment of a direction/arrow to each edge $e = (u, v) \in G$ such that e points either from u to v (denoted $u \to v$), or from v to u (denoted $v \to u$). We can think of this as 'drawing an arrow' one way or the other for e.

An **acyclic orientation** of a graph G is an orientation of G such that no cycle is formed by the edges. In other words, there does not exist a structure $v_1 \to v_2 \to \cdots \to v_n \to v_1$ for $n \geq 2$.

Observation Any acyclic orientation of a graph is a digraph (see Fig. 4 for an example).

Observation It is always possible to generate an acyclic orientation for any graph: one construction is to label the vertices with different numbers, and for any edge, draw arrows towards the larger number. Fig. 5 is an example, with arrows going to later letters in the alphabet.

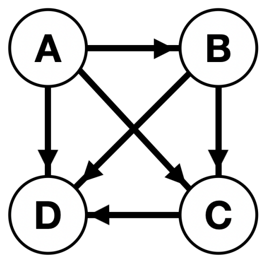

Figure 5: A labeled acyclic digraph on K_4, constructed using the algorithm in the preceding paragraph.

The two theorems I will prove are presented below. These are both results in [Sta06]. There are still a few definitions I will provide, such as $\bar{\chi}(\lambda)$, which will be done in individual theorem sections.

Theorem 1 [**Sta06, Theorem 1.2, Corollary 1.3**] For all non-negative integers λ, $\bar{\chi}(\lambda) = (-1)^{|V|}\chi(-\lambda)$. Importantly, this means $(-1)^{|V|}\chi_G(-1)$ is the number of acyclic orientations of G.

Theorem 2 [**Sta06, Proposition 2.1**] If $f(n)$ is the number of labeled acyclic directed graphs with n vertices, then

$$\sum_{n=0}^{\infty}\left(f(n)\frac{x^n}{n!}2^{\binom{n}{2}}\right) \cdot \sum_{n=0}^{\infty}\left((-1)^n\frac{x^n}{n!}2^{\binom{n}{2}}\right) = 1.$$

The paper is structured as follows. We start with background and motivation on the problem, then explain preliminary definitions in an introduction. Now we are at the end of the introduction. The third section will be a statement and proof of Theorem 1; the fourth section, a statement and proof of Theorem 2. A final section will conclude and explore related work.

3 Equivalence of $\bar{\chi}(\lambda)$ and acyclic orientations of a graph (Theorem 1)

Before we introduce the proof, let's finish the setup. We use a lemma which we addressed in Talk 2.

Lemma 1 [**Sta06, Proposition 1.1**] The chromatic polynomial $\chi_G(\lambda)$ equals the number of pairs (M, O), with M some coloring of the vertices of G and O an acyclic orientation of G, under the condition that if $u \to v$

in O, then $c_u > c_v$. Here c_i refers to the color number of vertex i.

Note: If (M, O) is a valid pair in Lemma 1, then M and O are said to be **compatible** with each other.

Color Example If we have $\lambda = 3$ colors, and blue is given value 3, green 2, and red 1, then if there is a blue vertex and a green vertex, in a valid pair (M, O), it is necessary to have the arrow point from the green vertex towards the blue vertex because $2 < 3$.

Example In Fig. 6, the chromatic polynomial for the graph is $\lambda(\lambda-1)^2(\lambda-2)$. One way (among many) to determine χ is to start at vertex 3, which has λ choices; then, vertex 4 has $\lambda - 1$ choices. If we go to vertex 2 next, there are $\lambda - 1$ options (because vertex 4 will reduce the color possibilities of vertex 2 by 1, but vertex 3, which is not adjacent to vertex 2, has no effect); then, vertex 1 has $\lambda - 2$ choices. Suppose we had $\lambda = 3$ colors, and blue was highest with value 3, green with value 2, and red with value 1. Then there are 12 possibilities, depicted in Fig. 7. This can be found in one of two ways: start with an acyclic orientation, and then 'fill in' the colors in all possible ways; or start with a coloring, and 'fit' arrows to correspond to the arrows. The latter is usually easier, because arrows are uniquely determined by what their colors are.

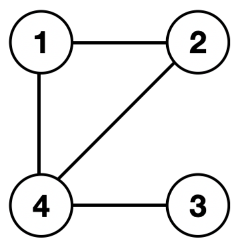

Figure 6: A small graph with 4 vertices and 4 edges.

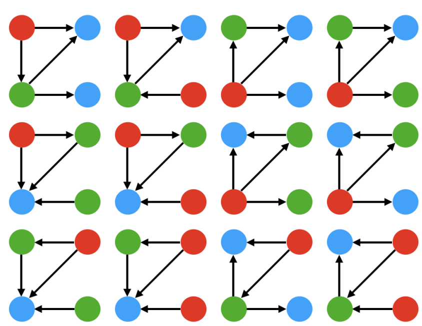

Figure 7: The 12 possible pairs (M, O) on the graph in Fig. 6, with each M some coloring of the vertices of the graph, and O an acyclic orientation of the graph. M and O are compatible with each other. Here $\lambda = 3$, with the colors blue $>$ green $>$ red.

Proof of Lemma 1. We use a bijection.

Every pair (M, O) corresponds to one correct coloring of the vertices of G: Suppose we have a pair (M, O). Then, M uniquely determines the coloring of the vertices of G, while O establishes the unique valid orientation of the edges on these colors (because we know no two adjacent vertices have the same color; thus, one color number is higher, and the arrow points away from it).

Every valid coloring of the vertices of G corresponds to one pair (M, O): Suppose we have a valid coloring. Then there must exist exactly one acyclic orientation O corresponding to this coloring, as for each edge, one color is of a higher color number than the other, and this dictates the direction that edge must point in. All edges are independent. ■

Lemma 1 motivates the following definition.

The **chromatic bar** $\bar{\chi}_G(\lambda)$ is the number of pairs (M, O) with the same definition as Lemma 1, except that $c_u \geq c_v$. In other words, the difference between this and Lemma 1's constraints is that here, we allow colors of neighboring vertices to be equal; this represents a relaxation of Lemma 1.

We also note some well-known properties on chromatic polynomials, as found in [Big93].

i) **Single Vertex Coloring Property**: For $G_0 =$ a graph with 1 vertex (and no edges), $\chi_{G_0}(\lambda) = \lambda$.

ii) **Disjoint Graph Coloring Property**: For any disjoint graphs G_1 and G_2, $\chi_{G_1 \sqcup G_2}(\lambda) = \chi_{G_1}(\lambda)\chi_{G_2}(\lambda)$.

iii) **Deletion/Contraction Property**: Given a graph G and any edge $e \in G$, we have $\chi_G(\lambda) = \chi_{G-e}(\lambda) - \chi_{G/e}(\lambda)$. $G-e$ (the deletion) is G without edge e, while G/e (the contraction) is G with e and its two neighboring vertices 'shrunk' into a point.

The three properties uniquely determine χ. In other words, if some polynomial f satisfies these three properties, then it must be the chromatic polynomial.

Before we continue, we give an example of the deletion/contraction property. Let $\lambda = 3$. Consider the small graph G we saw from Fig. 6, also left side of Fig. 8. Suppose the edge e between vertices 3 and 4 was contracted, so that the two vertices and e formed a single vertex C (see right graph in Fig. 8). It is not hard to see that the chromatic polynomial on G/e with $\lambda = 3$ equals 6: each of the $3! = 6$ color combinations on the vertices yields a valid acyclic orientation of G/e. Then, consider the graph $G - e$: the edge is removed. The chromatic polynomial on $G - e$ is $3! \cdot 3 = 18$, because the subgraph formed by the vertices $1, 2, 4$ is the identical scenario to what we just discussed with G/e, while vertex 3 can be any color, and given property ii) of χ, or the Disjoint Graph Coloring Property (i.e. independence of disjoint graphs), we have $6 \cdot 3$ possibilities. Putting this all together, we have $\chi_G(3) = 18 - 6 = 12$, which matches the answer given by our chromatic polynomial $\chi_G(\lambda) = \lambda(\lambda-1)^2(\lambda-2) = 3 \cdot 2^2 \cdot 1 = 12$.

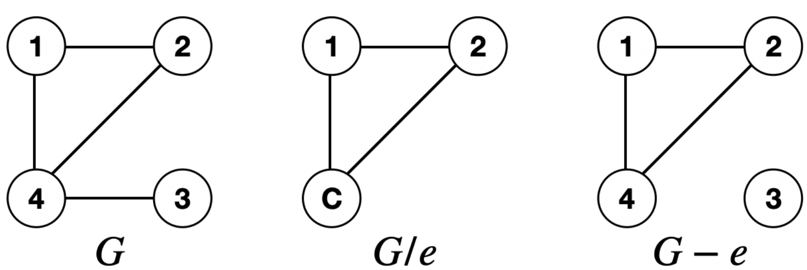

Figure 8: Left: The graph from Fig. 6, call it G. Middle: The contraction graph G/e, where e is the edge connecting vertices 3 and 3. The resulting contraction, which is a vertex, has been labeled C. Right: The deletion graph $G - e$.

Theorem 1 [**Sta06, Theorem 1.2**] For all non-negative integers λ, $\bar{\chi}(\lambda) = (-1)^{|V|}\chi(-\lambda)$.

Example Consider the graph from Fig. 6 (also seen as the left graph of Fig. 8), call it G. Let $\lambda = 1$ (let the single color be orange WLOG). We want to find the chromatic bar on G of $\lambda = 1$. Fig. 9 shows the enumeration of these graphs, and there are 12 of them. This matches with Theorem 1, which claims $\bar{\chi}(-1) = (-1)^4\chi(-1) = -1 \cdot (-2)^2 \cdot (-3) = 12$. In fact, the 12 possibilities here are just the acyclic orientations from Fig. 7, except all the vertices are colored orange.

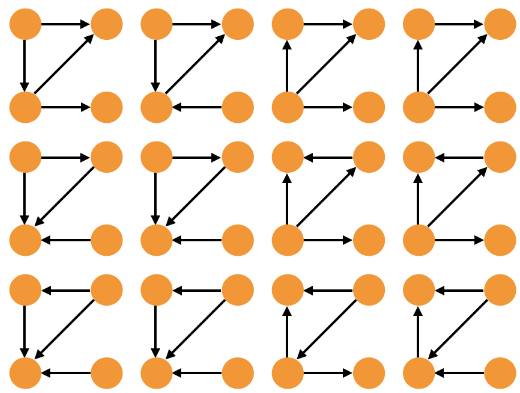

Figure 9: For $\lambda = 1$ on the graph in Fig. 6, the 12 valid acyclic orientations.

As will be found from the corollary after the proof of Theorem 1, it is this case $\lambda = 1$ we are most concerned about, as it equals the number of acyclic orientations on a graph. This relation also holds true for $\lambda = 2$ and higher values, but seeing that enumeration on $\lambda = 2$ will already yield $-2 \cdot (-3)^2 \cdot (-4) = 72$ possibilities, we forgo such an example.

Proof of Theorem 1 From our discussion of the three properties above, if we can prove $\bar{\chi}$ satisfies them, then $\bar{\chi}$ must be a chromatic polynomial.

The theorem states that $\bar{\chi}$ is exactly $\chi(-\lambda)$, with maybe a negative sign. This is to ensure that both sides of the equality are positive. Furthermore, observe that $\bar{\chi}$'s only difference from χ is the allowance of $c_u = c_v$. Therefore, if we can prove the same three properties for $\bar{\chi}$ as we did for the chromatic polynomial, we will be done. There is a slight change: Rather than a minus sign between the two terms in the third property, we use a plus sign. This is correct because we allow colors to be the same.

Property i): This is not hard to see, as we have one vertex and no edges, and given λ colors, G can be one of λ colors as well.

Property ii): This follows from independence of G_1 and G_2 as graphs.

The proof of property iii) is significantly longer and will take up the rest of the proof. Notation-wise, let the LHS (or LHS tally) be $\chi_G(\lambda)$, and let the RHS (or RHS tally) be $\chi_{G-e}(\lambda) - \chi_{G/e}(\lambda)$.

Property iii): First, take some arbitrary edge $e \in G$. Let $G' = G - e$ for ease of notation. Call the vertices adjacent to e v_1 and v_2. The graph of interest on the LHS $\bar{\chi}_G(\lambda)$ is composed of two separate cases: Either e goes $v_1 \to v_2$ in G (call this directed graph on G DG_r), or e goes $v_2 \to v_1$ (call this digraph DG_l). Let O be an arbitrary orientation on G' with (M, O) a valid pair in the chromatic bar. Let O_r be O including $v_1 \to v_2$, and O_l be O including $v_2 \to v_1$. An example of this configuration is introduced in Fig. 10.

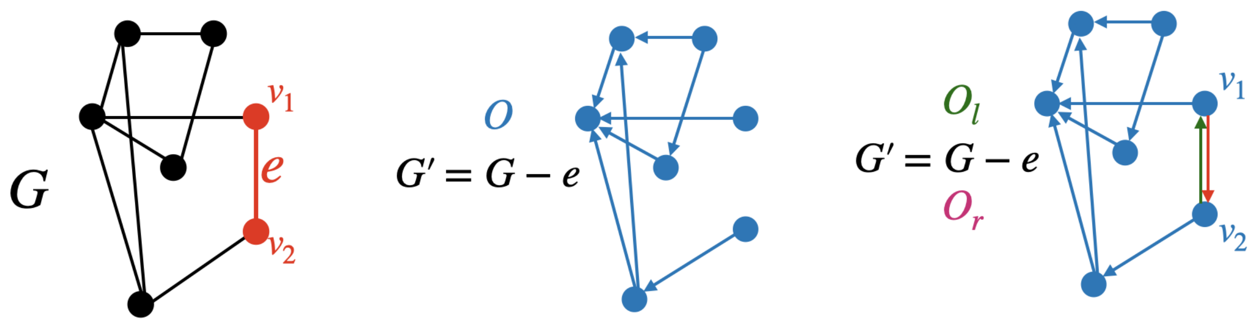

Figure 10: Left: A graph G, with a certain edge e selected. Center: The graph, with e removed. Right: The graph $G - e$, with the edge 'put back', either pointing from $v_2 \to v_1$ (as in the green edge of O_l) or vice versa (as in the red edge of O_r).

We now determine whether (M, O_l), (M, O_r), both, or neither are valid pairs in the chromatic bar. We are concerned with c_{v_1} and c_{v_2}.

To be more precise, given a valid pair (M, O) (corresponding to the LHS), we wish to count how many valid pairs among (M, O_l) and (M, O_r) exist (corresponding to the RHS, as we will demonstrate). Another way to think about this: We have a full graph without any edge removals in G on the LHS. Meanwhile, the RHS contains a term involving G' (G removed e), and another term involving a graph with contracted e. The counting of valid pairs among (M, O_l) and (M, O_r) helps determine $\bar{\chi}_{G'}(\lambda)$.

Suppose $c_{v_1} > c_{v_2}$. By the definition of the chromatic bar, (M, O_l) is invalid. It remains to prove that (M, O_r) is valid: we need to show that O_r is not cyclic (since O_l is invalid, we do not care about O_l's cyclicity). Indeed, suppose there existed a cycle $v_1 \to v_2 \to w_1 \to w_2 \to \cdots \to v_1$ in O_r. However, looking at their colors, this means $c_{v_1} > c_{v_2} \geq \cdots \geq c_{v_1}$, which is absurd. Thus, O_r is acyclic and we add 1 to our RHS tally.

Suppose $c_{v_2} > c_{v_1}$. Similarly to our above case, we obtain (M, O_r) is invalid, while O_l is acyclic and (M, O_l) is valid. We add 1 to our RHS tally.

Finally, suppose $c_{v_2} = c_{v_1}$. We claim that at least one of O_l, O_r is acyclic. Suppose that both were cyclic; then, O_l has a directed cycle $v_2 \to v_1 \to w_1 \to w_1 \to \cdots \to v_2$, while O_r has a directed cycle $v_1 \to v_2 \to x_1 \to x_2 \to \cdots \to v_1$. Combining the two, we see that a directed cycle exists from $v_1 \to w_1 \to w_2 \to \cdots \to v_2 \to x_1 \to x_2 \to \cdots \to v_1$, but this means O was not acyclic, which is a contradiction. This proves our claim. We add at least 1 to our RHS tally.

At this point, we see that the number of valid (M, O), which equals $\bar{\chi}_G(\lambda)$, is equal to $\bar{\chi}_{G'}(\lambda)$. Now, to include the addition of the $\bar{\chi}_{G/e}(\lambda)$, we will demonstrate that, for exactly $\bar{\chi}_{G/e}(\lambda)$ pairs of (M, O), the corresponding (M, O_l) and (M, O_r) are both valid (i.e. O_l and O_r are acyclic). These can only come from the final case when $c_{v_2} = c_{v_1}$, as it is the only one with "at least 1" added to our tally. Finishing this will give us the sum we require.

Critically, note that the edges of G' are equal to the edges of G/e (in both cases, e no longer exists; it is deleted and fused into v^+ along with v_1 and v_2, respectively). See Fig. 11 for a graphic depiction of this circumstance.

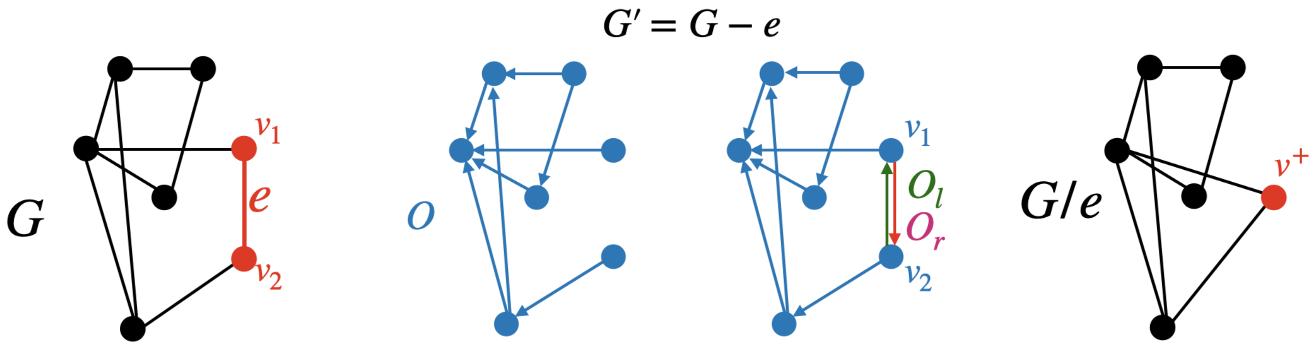

Figure 11: Left: The original graph G, with the edge e of concern. Middle graphs: The two possibilities for a directed edge for e. Right: The contraction G/e, with a new vertex v^+ replacing v_1, v_2, and e.

Now, consider some (M, O). We define a M' by $M'(v) = M(v)$ for all v in G', while $M'(v^+) = M(v_1) = M(v_2)$; in other words, all vertices in G have their original color. As for O', this is defined identically to O: the direction of each edge is preserved, and that's it because of the critical note above. It is not hard to see that we have a valid construction mechanism for both directions in F. Therefore, a bijection between each (M, O) with both O_l, O_r acyclic and (M, O_l) and (M, O_r) valid in the chromatic bar and each (M', O') valid for the chromatic bar on G/e has been established. As a result, we conclude that to the RHS, we can add an additional $\bar{\chi}_{G/e}(\lambda)$ to the tally. ■

At the end of the introduction we stated an important corollary: that, upon finishing the proof, $(-1)^{|V|}\chi_G(-1)$ is the number of acyclic orientations of G. We prove it below.

If we substitute $\lambda = 1$ into Theorem 1's equation, $\bar{\chi}(\lambda) = (-1)^{|V|}\chi(-\lambda)$, we obtain $\bar{\chi}(1) = (-1)^{|V|}\chi(-1)$. We claim that $\bar{\chi}(1)$ is the number of acyclic orientations of G. If we have an acyclic orientation, then coloring all of its vertices the same color (the only one allowed in $\lambda = 1$) is valid. If we color all vertices the same color, then any acyclic orientation works, as the equality case allowable under the chromatic bar holds. Therefore, $(-1)^{|V|}\chi(-1)$ equals the number of acyclic orientations of G. ■

4 Summation with number of labeled acyclic digraphs with n vertices

The theorem above finds a fast way of computing the number of acyclic orientations of a graph. If we can determine its chromatic polynomial, which can almost always be done in similar fashion as we did for K_4 - that is, an explicit closed form found for a graph - then we merely need to substitute $\lambda = -1$ and possibly correct for sign to find this quantity.

We also wish to find a quick way of finding the number of labeled acyclic digraphs with n vertices, which will be called $f(n)$. One way to think about the problem is to imagine n vertices, then construct some edges between the vertices, and give them each a direction, as in Fig. 12.

Observation There are n labeled vertices; thus, there are $\binom{n}{2}$ possible edges: each is either drawn in the graph, or not. This means there are $2^{\binom{n}{2}}$ different **non-directed** labeled graphs on these n vertices. Introducing an orientation on any graph (i.e. a direction to each edge) will non-strictly increase this number even further: for each non-directed graph, there exists at least one labeled acyclic digraph (consider all arrows pointing towards higher vertex number). We can say there is $o(2^{\binom{n}{2}})$ labeled acyclic digraphs on n vertices.

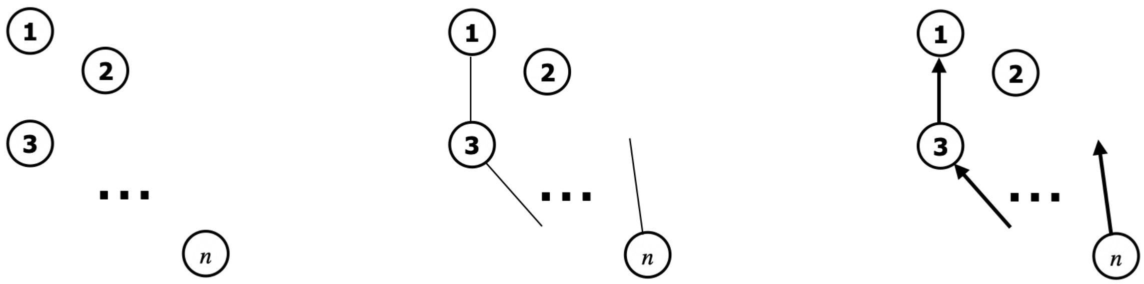

Figure 12: Left: n vertices in space. Center: n vertices with some edges between them selected. Right: Assigning a direction to those chosen edges.

Theorem 2 [**Sta06, Proposition 2.1**] If $f(n)$ is the number of labeled acyclic directed graphs with n vertices, then

$$\sum_{n=0}^{\infty}\left(f(n)\frac{x^n}{n!}2^{\binom{n}{2}}\right)\cdot\sum_{n=0}^{\infty}\left((-1)^n\frac{x^n}{n!}2^{\binom{n}{2}}\right)=1.$$

The proof derives mostly not from [Sta06], but from [Rea60] and [BG71].

The equality does not determine $f(n)$ for any n directly. Instead, it sheds light on the behavior of f.

Proof of Theorem 2. First, observe by the extension on our proof of Theorem 1 that

$$f(n)=(-1)^n\sum_{G}\chi(G,-1)$$

where G represents one of the $2^{\binom{n}{2}}$ possible graphs for that choice of n.

Rather than consider the quantity $\sum_G \chi(G,-1)$, we consider $M_n(k)=\sum_G \chi(G,k)$, with k representing the number of colors allowed. Here k serves the same function as λ; the convention in all three papers mentioned above is to use k.

Suppose there were n_i nodes which had the color i in G, with $1 \le i \le k$. Let the partition of nodes into colors for a particular configuration be denoted $P_{n\alpha}$. We have $\sum n_i = k$. Furthermore, consider edges e_{ab}, edges that go between nodes with color a and b (no specificity in arrow direction). Let $E=\sum_{a<b} e_{ab}$, where we need the $a<b$ constraint because we are not giving direction to edges between colors a and b. On a high level, we first determine the colors of the nodes, and then we decide where to put edges given we must have a fixed number of edges, as well as a fixed number of edges going between any 'pair' of colors.

To give nodes their colors so that there are n_i nodes for color i, this has $n!\cdot\left(\prod_i n_i!\right)^{-1}$ ways. Now, for the edges, we need to have e_{ab} edges, and we can draw edges between $n_a n_b$ sets of nodes. Thus, we have $\binom{n_a n_b}{e_{ab}}$ ways to do this for each e_{ab}.

Putting this all together, for a certain value of E, we have

$$n! \cdot \left(\prod_i n_i!\right)^{-1} \cdot \prod_{a<b} \binom{n_a n_b}{e_{ab}}$$

choices. Now, we must add this over all $0 \le E \le \binom{n}{2}$. For a shortcut, observe that $\binom{n_a n_b}{e_{ab}}$ is, using a generating function, the coefficient of $x^{e_{ab}}$ on $(1+x)^{n_a n_b}$. Therefore, when we sum over E, we are finding

$$n! \cdot \left(\prod_i n_i!\right)^{-1} \cdot \prod_{a<b}(1+x)^{n_a n_b} = n! \cdot \left(\prod_i n_i!\right)^{-1} \cdot (1+x)^{\sum n_a n_b}$$

by turning the product up into a summation on the exponent.

Observe that $\sum n_a n_b$ requires a and b to be different; therefore, it is equal to $0.5((\sum n_i)^2 - \sum n_i^2)$. As a result, we can change the rightmost term in the product (the one with base $1+x$) to $(1+x)^{0.5((\sum n_i)^2 - \sum n_i^2)}$; as we sum over E, this will yield

$$n! \cdot \left(\prod_i n_i!\right)^{-1} \cdot \binom{0.5n^2 - 0.5\sum n_i^2}{E}.$$

However, this is an unwieldy formula. In particular, we would need to sum that expression over all possible partitions of nodes into colors $P_{n\alpha}$. This is difficult, so we relax the edges constraint. Instead, suppose we just considered k-colored graphs which can be created with partition $P_{n\alpha}$. Then we take this expression and sum it up over all E, which, by the binomial coefficient sum formula $\sum_{i=0}^{n} \binom{a}{i} = 2^a$, gives

$$n! \cdot \left(\prod_i n_i!\right)^{-1} \cdot 2^{0.5n^2 - 0.5\sum n_i^2}$$

Now, we can sum up over n to find $M_n(k)$, which equals

$$\sum_n \left(n! \cdot \left(\prod_i n_i!\right)^{-1} \cdot 2^{0.5n^2 - 0.5\sum n_i^2}\right)$$

We can apply the coefficient argument one more time on this quantity and realize that $M_n(k)$ equals $n! \cdot 2^{0.5n^2}$ multiplied by the coefficient of x^n in the summation $\left(\sum_{i=0}^{\infty} x^i \cdot (i!)^{-1} \cdot 2^{-0.5i^2}\right)^k$.

This equivalence gives us the full formation of the entire equality:

$$\sum_{n=0}^{\infty} 2^{-0.5n^2} M_n(k) \frac{x^n}{n!} = \left(\sum_{n=0}^{\infty} x^n \cdot (n!)^{-1} \cdot 2^{-0.5n^2}\right)^k$$

We make a switch of variables. Consider $x' = \sqrt{2}x$. Then, our term x^n equals $(x')^n \cdot 2^{-0.5n}$. This allows us to 'reduce' the exponent on the 2 on both sides from $-0.5n^2$ to the desired $\binom{n}{2}$, and a final summation over $k = 0$ to n with some simplification will lead us to the desired result. ■

5 Conclusion

In this paper we have demonstrated two theorems for calculation of niche but important properties on general graphs. These two are some of the earlier results in Stanley's famous landmark paper [Sta95]. Later, symmetric functions are used for generalizing chromatic polynomials and their properties. This concludes on a wider discussion for negative numbers on chromatic polynomials. Some conjectures and open directions relating equalities of the nature discussed in this paper revolve around chromatic symmetric functions and their varieties. The applications are wide, from analyzing Dyck paths in [CMP23] to working with matroids in [NZ21]. Another direction is Stanley's claw contractibility, introduced late in [Sta95]; it is analyzed in [RS03] and discusses a closure operation on claw-free graphs. Claw contractibility is an extension of a series of equalities, including Theorems 1 and 2, constructed in [Sta95].

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